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A TRANSPORTATION PROBLEM INVOLVING SOURCE-LOCATION OPTIMIZATION

by

R. L. Sielken Jr.

Texas A&M Research Foundation
Office of Naval Research
Contract N00014-68-A-0140
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ATTACHMENT II

A TRANSPORTATION PROBLEM INVOLVING

SOURCE-LOCATION OPTIMIZATION

R. L. Sielken Jr.

A generalization of the capacitated plant-location problem is formulated and several possible modifications noted. Four solution techniques are briefly discussed: exhaustive enumeration, probabilistic search, zero-one mixed integer linear programming, and an iterative procedure. The relative attractiveness of the iterative procedure is illustrated in three examples.

√

1. Introduction

The general source-location problem under consideration is how to supply J destinations with $D_1, D_2, \ldots, D_J$ units from K possible sources at a minimum cost when the K sources have capacities $B_1, B_2, \ldots, B_K$, and any subset of the K sources can be located at any one of I locations. A minimal cost solution involves the specification of each source's location and the allocation of the demands $D_1, D_2, \ldots, D_J$ among the sources.

A zero-one mixed integer linear programming formulation of this problem is:

$$\min \left[\Sigma_{k=1}^{K} \Sigma_{i=1}^{I} \Sigma_{j=1}^{J} c_{kij}\, x_{kij} + \sum_{k=1}^{K} \sum_{i=1}^{I} u_{ki}\, f_{ki} \right] \tag{1}$$

subject to the constraints

$$\Sigma_{k=1}^{K} \Sigma_{i=1}^{I} x_{kij} = D_j, \quad j = 1, \ldots, J, \tag{2}$$

$$\Sigma_{j=1}^{J} x_{kij} \leq u_{ki} B_k, \quad i = 1, \ldots I \text{ and } k = 1, \ldots, K, \tag{3}$$

$$\Sigma_{i=1}^{I} u_{ki} \leq 1, \quad k = 1, \ldots, K, \tag{4}$$

$$u_{ki} = 0 \text{ or } 1, \quad i = 1, \ldots, I \text{ and } k = 1, \ldots, K, \tag{5}$$

$$x_{kij} \geq 0, \quad \text{all } k, i, j, \tag{6}$$

where

$$u_{ki} = \begin{cases} 1 \text{ if source } k \text{ is at location } i, \\ 0 \text{ otherwise;} \end{cases}$$

f_{ki} = the fixed cost for source k being at location i;

x_{kij} = the number of units transported by source k from location i to destination j; and

c_{kij} = the cost of source k producing one unit at location i and transporting it to destination j.

The equality in (2) implies that the demand at each destination is satisfied. The constraint in (3) insures that, if source k is at location i, then the capacity of source k is not exceeded, and, if source k is not at location i, then nothing is transported by source k from location i. The inequality in (4) implies that each source is located in at most one location. The constraints also imply that, if a source is not located anywhere at all, then it cannot help satisfy the demands.

Of course, the physical interpretation of the sources, destinations, costs, etc. is quite unrestricted. For example, the sources could be refineries, the destinations could be military installations, and the costs in terms of time. Alternatively, the sources could be generators of electricity, the destinations cities, and the costs monetary.

2. Some Possible Modifications of the General Source-Location Problem.

Restrictions concerning the feasibility of certain source-location combinations can be easily incorporated. For example, if at most L_i sources can be located at location i, then the problem should be modified by adding the constraint

$$\sum_{k=1}^{K} u_{ki} \leq L_i \,. \tag{7}$$

Of course, if source k can only be located at a location in a subset T_k of the I locations, then the problem should be modified by deleting all variables u_{ki}

and x_{kij} with $i \notin I_k$

The extension of the problem to a situation involving more than one product is conceptually simple. If there are P products and a subscript p is used to indicate the p-th product, the extended problem is

$$\min \Sigma_{p=1}^{P} \Sigma_{k=1}^{K} \Sigma_{i=1}^{I} \Sigma_{j=1}^{J} c_{kijp} x_{kijp} + \sum_{k=1}^{K} \sum_{i=1}^{I} u_{ki} f_{ki}, \tag{8}$$

subject to the constraints

$$\Sigma_{k=1}^{K} \Sigma_{i=1}^{I} x_{kijp} = D_{jp}, \text{ all } j, p, \tag{9}$$

$$\Sigma_{j=1}^{J} x_{kijp} \leq u_{ki} B_{kp}, \text{ all } k, i, p, \tag{10}$$

$$\Sigma_{i=1}^{I} u_{ki} \leq 1, \quad k = 1, \ldots, K, \tag{11}$$

$$u_{ki} = 0 \text{ or } 1, \text{ all } k, i, \tag{12}$$

$$x_{kijp} \geq 0, \text{ all } k, i, j, p. \tag{13}$$

Of course, the general source-location problem as formulated in (1)-(6) encompasses the special case in which all K sources are identical. However, if all K sources are identical, the problem can also be formulated as

$$\min \Sigma_{i=1}^{I} \Sigma_{j=1}^{J} c_{ij} x_{ij} + \sum_{i=1}^{I} v_i f_i \tag{14}$$

subject to the constraints

$$\Sigma_{i=1}^{I} x_{ij} = D_j, \quad j = 1, \ldots, J, \tag{15}$$

$$\Sigma_{j=1}^{J} x_{ij} \leq v_i B, \quad i = 1, \ldots, I, \tag{16}$$

$$\Sigma_{i=1}^{I} v_i \leq K, \tag{17}$$

$$v_i = 0, 1, 2, \ldots, K, \quad i = 1, \ldots, I. \tag{18}$$

$$x_{ij} \geq 0, \text{ all } i, j, \tag{19}$$

where

B = the capacity of a source;

c_{ij} = the cost of producing one unit at location i and transporting it to destination j;

x_{ij} = the number of units produced at location i and transported to destination j;

f_i = the fixed cost per source located at location i; and

v_i = the number of sources at location i.

This alternative formulation suggests that for this particular case specialized solution techniques should be used. The development of such specialized techniques is in progress at this time.

3. A Survey of the Literature

The "capacitated plant-location problem" considered by Bulfin and Unger [3], Davis [9], Ellwein [12], Gray [14], [15], and [16], Marks [19], Sá [21], and Spielberg [24] can be formulated as

$$\min \ \Sigma_{k=1}^{K*} \ \Sigma_{j=1}^{J} c_{kj} x_{kj} + \Sigma_{k=1}^{K*} f_k u_k \tag{20}$$

subject to the constraints

$$\Sigma_{k=1}^{K*} \ x_{kj} = D_j, \quad j = 1, \ldots, J \tag{21}$$

$$\Sigma_{j=1}^{J} \ x_{kj} \leq u_k B_k, \quad k = 1, \ldots, K* \tag{22}$$

$$u_k = 0 \text{ or } 1, \quad k = 1, \ldots, K* \tag{23}$$

$$x_{kj} \geq 0, \quad \text{all } k, j . \tag{24}$$

Thus, from one viewpoint, the capacitated plant-location problem is a general source-location problem with $I = 1$. Alternatively, from the opposite viewpoint, the general source-location problem is a capacitated plant-location problem with K sets of I mutually exclusive plants. Some discussion of mutually exclusive plants is given by Marks [19].

The "simple plant-location problem" considered by Celebiler [4], Efroymson and Ray [11], Feldman, Lehrer, and Ray [13], Manne [18], and Spielberg [22] and [23] has the same formulation as the capacitated plant-location problem except that the capacity of each source is assumed to be at least equal to the total

demand. This assumption simplifies the problem considerably since it implies that each demand D_j can always be optimally supplied from the "nearest" source.

Warehouse location problems which can be simplified to general source-location problems were considered by Balinski and Mills [1], Baumol and Wolfe [2], Kuehn and Hamburger [17], and Marks [19].

Chapelle [5], Cooper [6], [7], and [8], and Rainco3ek and Hartley [20] investigated the following source-location problem: If

D_j = the demand for a product at destination j, $j = 1, \ldots, J$;

(a_{Dj}, b_{Dj}) = the coordinates in two-dimensional Euclidean space of destination j;

B_k = the capacity of source k, $k = 1, \ldots, K$;

(a_k, b_k) = the coordinates in two-dimensional Euclidean space of source k; and

x_{kj} = the number of units transported from source k to destination j;

then the objective is to

$$\min \ \Sigma_{k=1}^{K} \ \Sigma_{j=1}^{J} x_{kj} \, [(a_{Dj} - a_k)^2 + (b_{Dj} - b_k)^2]^{1/2} \tag{25}$$

subject to the constraints

$$\Sigma_{k=1}^{K} x_{kj} = D_j, \quad j = 1, \ldots, J, \tag{26}$$

$$\Sigma_{j=1}^{J} x_{kj} \leq B_k, \quad k = 1, \ldots, K, \tag{27}$$

$$a_k \geq 0, \; b_k \geq 0, \; x_{kj} \geq 0, \quad \text{all } k, j. \tag{28}$$

This problem is the same as the general source-location problem except that

(i) the location of each of the K rces is not restricted to one of a finite number of possibilities, and

(ii) c_{kij}, the cost of source k producing one unit at location i and transporting it to destination j, has been replaced by the Euclidean distance from the location of source k to destination j.

4. Some Solution Techniques

4.1 Exhaustive Enumeration

To solve the general source-location problem by exhaustive enumeration, the optimal allocation of the J demandsamong the K sources would have to be determined for each of the I^K possible assignments of the K sources to the I locations. For each assignment the determination of the demand allocation is a capacitated plant-location problem; that is

$$\min \Sigma_{k=1}^{K} \Sigma_{j=1}^{J} c_{kj} x_{kj} + \Sigma_{k=1}^{K} f_k u_k \tag{29}$$

subject to the constraints

$$\Sigma_{k=1}^{K} x_{kj} = D_j, \quad j = 1, \ldots, J, \tag{30}$$

$$\Sigma_{j=1}^{J} x_{kj} \leq B_k u_k, \quad k = 1, \ldots, K, \tag{31}$$

$$u_k = 0 \text{ or } 1, \quad k = 1, \ldots, K \tag{32}$$

$$x_{kj} \geq 0, \quad \text{all } k, j. \tag{33}$$

Branch-and-bound treatments of this problem have been given in [3], [9], [12], [21] and [24]. The empirical evidence suggests that the branch-and-bound method would be most useful in small problems, around 25 integer variables or less.

There are at least two difficulties with the exhaustive enumeration approach:

(i) the number of possible assignments, I^K, increases dramatically as I and K increase, and

(ii) the only available "fast" algorithms for solving medium or large capacitated plant-location problems are approximate routines - see, for example, [21].

4.2 Probabilistic Search

Since the number of possible source-location assignments may be quite large in a realistic situation, it may only be feasible to explicitly evaluate a subset of them. A simple probabilistic search procedure which explicitly considers only a subset of the possible source-location assignments is as follows. Select a random sample of size n without replacement from the set of all possible assignments. Solve the n problems (29)-(33) corresponding to n assignments selected. Let $z_1^*, z_2^*, \ldots, z_n^*$ represent the optimal values of the objective functions in these problems. Then, if $z_{(1)}, z_{(2)}, \ldots, z_{(I^K)}$ are the order values of $z_1, z_2, \ldots, z_{I^K}$ with

$$z_{(i)} \leq z_{(i+1)}, \quad i = 1, \ldots, I^K-1, \tag{34}$$

it follows that

$$P\left(\min_{1 \leq m \leq n} z_m^* \leq z_{(r)}\right) \geq 1 - \binom{I^K-r}{n} \Big/ \binom{I^K}{n}$$
$$= 1 - \frac{(I^K-r)(I^K-r-1) \cdots (I^K-r-n+1)}{I^K(I^K-1) \cdots (I^K-n+1)} . \tag{35}$$

Thus, for any r, the probability of obtaining an assignment at least as good as the r-th best assignment can be made as close to one as desired by taking the sample size n sufficiently large.

One procedure for sequentially determing which source-location assignements are to be evaluated is as follows. Let α be a constant such that $0 < \alpha < 1$. Let $z_1, z_2, \ldots, z_{I^K}$ represent the optimal values of the objective functions in the I^K capacitated plant-location problems (29)-(33) corresponding to the I^K possible assignments. Assume that a histogram of $z_1, \ldots, z_{I^K}$ can be closely approximated by a density function of a given form. Take a random sample of size one from the set of I^K assignments. Solve the problem in (29)-(33) corresponding to the assignment selected. Estimate the parameters of the approximating density function. Then repeat these three steps until a selected assignment makes (29) less than the α-th percentile of the distribution function corresponding to the estimated density. This procedure was discussed by Cooper [7] in relation to a similar source-location problem.

In comparing these two probabilistic search procedures, the former procedure has the advantage of selecting a fixed number of source location assignments for consideration and does not necessitate any assumption about the distribution of $z_1, z_2, \ldots, z_{I^K}$. On the other hand, Cooper's sequential procedure could result in a smaller sample size.

4.3 Zero-One Mixed Integer Linear Programming

The general source-location problem is a zero-one mixed integer linear programming problem; hence, any general mixed integer linear programming algorithm may theoretically be used to solve it. In particular, the relatively new branch-and-bound methods and implicit enumeration methods seem applicable. However, computational experience on the simpler capacitated plant-location problems indicates that these methods may only be computationally feasible for relatively small problems.

4.4 An Iterative Algorithm

If the source-location variables, u_{ki}, were known, then the general source-location problem would reduce to

$$\min \Sigma_{k=1}^{K} \Sigma_{j=1}^{J} \Sigma_{i=1}^{I} c_{kij} x_{kij} \tag{36}$$

subject to the constraints

$$\Sigma_{k=1}^{K} \Sigma_{i=1}^{I} x_{kij} = D_j, \quad j = 1, \ldots, J + 1, \tag{37}$$

$$\Sigma_{j=1}^{J} x_{kij} = u_{ki} B_k, \quad \text{all } k, i, \tag{38}$$

$$x_{kij} \geq 0, \quad \text{all } k, i, j, \tag{39}$$

where

$$D_{J+1} = \Sigma_{k=1}^{K} \Sigma_{i=1}^{I} u_{ki} B_k - \Sigma_{j=1}^{J} D_j. \tag{40}$$

This problem is just a simple transportation problem.

On the other hand, if the allocations of the demands $D_1, \ldots, D_J$ among the K sources were known and

D_{kj} = the number of units to be transported from source k to destination j, then the general source-location problem would reduce to

$$\min \sum_{k=1}^{K} \sum_{i=1}^{I} u_{ki} \left[f_{ki} + \sum_{j=1}^{J} c_{kij} D_{kj}\right] \tag{41}$$

subject to the constraints

$$\Sigma_{i=1}^{I} u_{ki} = 1, \quad \text{for all k such that } \Sigma_{j=1}^{J} D_{kj} > 0. \tag{42}$$

$$u_{ki} = 0 \text{ or } 1, \quad \text{all k, i.} \tag{43}$$

An optimal solution to this problem is simply

$$u_{ki} = 1 \text{ if } i = i^*(k) \text{ and } \Sigma_{j=1}^{J} D_{kj} > 0 \tag{44}$$

$$= 0 \text{ otherwise}$$

where, for each k, $i^*(k)$ is the smallest positive integer i' such that

$$f_{ki'} + \sum_{j=1}^{J} c_{ki'j} D_{kj} = \min_{1 \leq i \leq I} \left[f_{ki} + \sum_{j=1}^{J} c_{kij} D_{kj}\right]. \tag{45}$$

The simplicity of both optimally allocating demands among sources for a fixed source-location configuration and determining an optimal source-location configuration for a fixed allocation of demands among sources suggests the following iterative

scheme for obtaining approximate solutions to the general source-location problem:

(i) Choose an initial source-location configuration.

(ii) For this source-location configuration, determine the optimal allocation of demands among the sources.

(iii) For this demand allocation, determine the optimal source-location configuration. Return to (ii) with this new configuration.

This scheme of alternately evaluating optimal demand allocations and optimal source-location configurations can be formalized into the following procedure:

1. Select an initial value for each u_{ki} with

$$u_{ki} = 0 \text{ or } 1, \quad \text{all } k, i$$

and

$$\sum_{i=1}^{I} u_{ki} \leq 1, \quad k = 1, \ldots, K.$$

2. Solve the transportation problem in (36)-(40) with the u_{ki}'s fixed at their current values. Represent the optimal demand allocation thus obtained by $\{D_{kj};\ k = 1, \ldots, K \text{ and } j = 1, \ldots, J\}$ where D_{kj} is the number of units to be transported from source k to destination j; ie,

$$D_{kj} = \sum_{i=1}^{I} x_{kij}.$$

3. Generate new u_{ki} values in accordance with (44), so that the corresponding source-location configuration is optimal for the new demand allocation determined in step 2.

4. If the source-location configuration determined in step 3 is the same configuration that was last used in step 2, the iterative procedure terminates, and the optimal source-location configuration and demand allocation are approximated by the source-location configuration and demand allocation corresponding to the current u_{ki}'s and D_{kj}'s respectively. On the other hand, if the source-location configuration has changed, return to step 2 with the current u_{ki} values being those just determined in step 3

An extremely attractive feature of this iterative procedure is that, if the objective function for the general source-location problem,

$$\Sigma_{k=1}^{K} \ \Sigma_{i=1}^{I} \ \Sigma_{j=1}^{J} c_{kij} \ x_{kij} + \Sigma_{k=1}^{K} \ \Sigma_{i=1}^{I} f_{ki} \ u_{ki}, \tag{46}$$

was calculated each time step 2 was completed and each time step 3 was completed, the result would be a nonincreasing sequence. This characteristic follows immediately since in step 2 the previous demand allocation is feasible when the new optimal demand allocation is determined and in step 3 the previous source-location configuration is feasible when the new optimal configuration is determined. Thus, each time step 2 is completed and each time step 3 is completed an improved feasible solution to the general source-location problem is obtained or a feasible solution which is at least as good as its predecessor. This feature is quite important for very large problems in which a search for the optimal solution is often economically impossible.

Since there are only $(I+1)^K$ possible source-location configurations, the iterative procedure will terminate in a finite number of steps provided no

source-location configuration is generated in step 3 infinitely often. In step 2, the value of (46), the objective function for the general source-location problem, is minimized for the current source-location configuration. Thus, since the sequence of values of (46) determined by the iterative procedure is a non-increasing sequence, the only way in which a source-location configuration could be generated in step 3 infinitely often is for there to be more than one source-location configuration with the same minimum value of (46) and for looping or cycling to occur among these configurations. However, such looping can be easily dealt with. For instance, in the examples to follow the stopping rule in step 4 was augmented so that termination would occur if the same value of (46) occurred more than four times. Interestingly, the iterative procedure was never terminated for this reason.

Since this iterative procedure yields only a locally optimal solution as opposed to a necessarily globally optimal solution, it will generally be worthwhile to repeat the iterative procedure with different initial source-location configurations.

5. Examples of the Iterative Procedure's Performance

In the previous section several attractive characteristics of the iterative procedure were indicated. To further substantiate this attractiveness, the iterative procedure was tried on three sample problems involving real data. The parameters and problem characteristics are indicated in Table 1.

In Examples 1 and 2 the iterative procedure was carried out for each of 100 initial source-location configurations selected at random. Since Example 3 corresponds to a much larger sample problem, monetary considerations allowed

only 33 random initial source-location configurations to be considered.

The cost (46) associated with the optimal demand allocation for the initial source-location configuration and the cost associated with the optimal demand allocation for the source-location configuration determined to be approximately optimal by the iterative procedure were computed for each of the initial source-location configurations. As hoped, the costs associated with the approximate solutions generated by the iterative procedure were much smaller than the costs associated with the optimal demand allocations for the random initial source-location configurations. The magnitude of this improvment is indicated in Table 2.

The effort involved in the iterative procedure's determination of an approximate solution is reflected by the number of source-location configurations for which the corresponding optimal demand allocation had to be determined. These numbers which include the initial source-location configurations were surpriseingly small. As shown by the summary of these numbers in Table 3, the iterative procedure did not just move gradually toward the near optimal solutions but rather leaped toward them.

For comparative purposes, the zero-one mixed integer linear programming problem corresponding to the sample problem in Example 1 was also solved using a branch-and-bound algorithm. This algorithm was initially programmed by Westphal and Gately [26] who based the algorithm on the branch-and-bound procedure described by Davis, Kendrick, and Weitzman [10]. The algorithm was modified to incorporate the improvements in the bounding procedure which were suggested by Tomlin [23].

The optimal solution has a value of 91 which is also the value of the best approximate solution determined by the iterative procedure. The branch-and-bound algorithm required 85 minutes of IBM 360/65 central processing time to determine the optimal solution.

The optimal demand allocation for a given source-location configuration was determined by an out-of-kilter network-flow algorithm described by Clasen [27]. The total number of IBM 360/65 central processing minutes required for all of the trials of the iterative procedure in Example 1, 2, and 3 was only 12, 12, and 20 respectively.

6. Evaluation of the Iterative Procedure

The performance characteristics of the iterative procedure in the three sample problems combined with the attractiveness of its theoretical and practical properties as described in section 4.4 makes the iterative procedure an extremely attractive practical tool for determining near optimal solutions to general source-location problems.

TABLE 1. Parameters for Examples 1, 2, and 3

Parameter	Interpretation	Definition		
		Example 1	Example 2	Example 3
K	Number of sources	4	4	10
I	Number of locations	8 largest U.S. metropolitan areas	8 largest U.S. metropolitan areas	30 largest cities in Texas
J	Number of destinations	16 largest U.S. metropolitan areas	16 largest U.S. metropolitan areas	30 largest cities in Texas
c_{kij}	Cost for source k to produce one unit at location i and transport it to destination j	Proportional to highway mileage	Proportional to highway mileage	Proportional to highway mileage
f_{ki}	Fixed cost for source k being at location i	Proportional to capacity of source k and cost of living index for i-th metropolitan area	Proportional to capacity of source k and cost of living index for i-th metropolitan area	The same for all k, i
D_j	Demand at destination j			
B_k	Capacity of source k	B_1 = .3 x total demand B_2 = .5 x total demand B_3 = .7 x total demand B_4 = 1.0 x total demand	.5 x total demand for k = 1, ..., 4	.2 x total demand for k = 1, ..., 10

TABLE 2. The Costs of the Optimal Demand Allocations for the Initial Source-Location Configuration and the Iterative Procedure's Approximately Optimal Source-Location Configuration.

n	n-th Smallest Cost					
	Example 1		Example 2		Example 3	
	Initial Configuration	"Optimal" Configuration	Initial Configuration	"Optimal" Configuration	Initial Configuration	"Optimal" Configuration
1	91	91	91	91	347	219
2	96	91	95	91	362	220
3	98	91	98	91	402	224
4	104	92	98	91	412	224
5	107	92	100	91	427	226
10	111	92	111	91	474	234
15	114	95	112	92	520	259
20	116	95	116	92	553	306
25	119	95	116	95	624	316
30	121	96	118	95	759	343
35	124	97	119	95		
40	128	97	124	96		
45	133	100	127	96		
50	135	100	133	96		
60	144	107	145	100		
70	165	109	152	105		
80	194	111	194	108		
90	281	115	250	111		
100	518	126	518	115		

TABLE 3. The Number of Source-Location Configurations for which the Optimal Demand Allocation was Determined Enroute to an Approximately Optimal Source-Location Configuration.

Number *	Frequency		
	Example 1	Example 2	Example 3
1	3	2	0
2	73	72	10
3	22	24	17
4	2	2	5
5	0	0	1

*This count includes the initial source-location configuration.

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